Hence to first order

$$\ddot{\lambda} + \lambda \frac{1 + 4E \cos \theta}{1 + E \cos \theta} = G_0 r_0 K^{-2} + 2\int G_0 dr_0 K^{-2} + \text{terms in } \nu,$$

[of which a solution was published in the *Proceedings of the London Mathematical Society*, 1923 February 8:—

"The solution of the equation

$$d/d\theta[d^2\lambda/d\theta^2 + \lambda(1 + 4E \cos \theta)/(1 + E \cos \theta)] = d\xi/d\theta + 2\eta,$$

in which E is constant, may be written in the form

$$\begin{aligned}\lambda = S(1 + EC)[\lambda_1 + \int C(1 + EC)^{-1}\xi d\theta] \\ + C(1 + EC)\lambda_2/2E + C^2(2 + EC)(1 + EC)^{-1}\int \eta d\theta \\ - C(1 + EC)^{-1}[\lambda_3 + \int\{C(2 + EC)\eta + S(1 + EC)\xi\}d\theta] \\ + 2S(1 + EC)\int C(1 + EC)^{-3}[E\lambda_3 + \int\{(1 + EC)^2\eta + ES(1 + EC)\xi\}d\theta]d\theta,\end{aligned}$$

in which $C = \cos \theta$ and $S = \sin \theta$, and λ_1, λ_2, λ_3 are constants."]

Lastly, in regard to ν, Laplace showed that $\dot{\nu}$ contained no *secular* terms of the first and second orders as to perturbing masses, but the reference was not to small planets.

A Method of Extrapolation of Perturbations. By B. V. Noumerov.

(*Communicated by the Secretaries.*)

I have pointed out in my previous work * the advantage which is to be got in the numerical integration of the equations of perturbed motion by the introduction of special rectangular co-ordinates. In fact, if we denote by $\bar{x}$, $\bar{y}$, $\bar{z}$ the special co-ordinates connected with ordinary heliocentric rectangular co-ordinates by the equations

$$\bar{x} = x\left(1 + \frac{k^2\omega^2}{12r^3}\right); \quad \bar{y} = y\left(1 + \frac{k^2\omega^2}{12r^3}\right); \quad \bar{z} = z\left(1 + \frac{k^2\omega^2}{12r^3}\right) \quad . \quad (1)$$

where k is the Gaussian constant, ω the interval of integration, and r the radius vector, the fundamental formula of extrapolation (for $\bar{x}$) in the case of perturbed motion is

$$\bar{x}_3 = \bar{x}_2(2 - \sigma_2) - \bar{x}_1 + R_2 + \tfrac{1}{12}\Delta^2 R_2 + f \quad . \quad . \quad (2)$$

* B. Noumerov, "Méthode nouvelle de la détermination des orbites et le calcul des éphémérides en tenant compte des perturbations," *Publications de l'Observatoire Astrophysique Central de Russie*, vol. ii., Moscou, 1923.

where σ is determined by the equation

$$\sigma\left(1-\frac{\sigma}{12}\right)^2=\frac{k^2\omega^2}{\bar{r}^3} \quad . \quad . \quad . \quad . \quad (3)$$

as a function of $\bar{r}^2=\bar{x}^2+\bar{y}^2+\bar{z}^2$; R is the effect of a perturbing planet with the co-ordinates x', y', z' and the mass m', and is given by

$$\mathrm{R}=k^2\omega^2m'\left(\frac{x'-x}{\rho^3}-\frac{x'}{r'^3}\right) \quad . \quad . \quad . \quad . \quad (4)$$

$\Delta^2\mathrm{R}_2$ is the second difference of R at the second moment, and finally f is the correction for higher derivatives or differences beginning with the 6th order.

$$f=-\tfrac{1}{240}\Delta^6+\tfrac{31}{60480}\Delta^8+\ldots \quad . \quad . \quad (5)$$

If the interval of integration is rightly chosen, it is permissible to neglect the effect of these higher-order terms in formula (2). Subsidiary tables helping to find the interval of integration as well as for determining σ from the equation (3) are given in the above-mentioned work. Computing the co-ordinates with six decimals, it is possible to carry on the integration with intervals of twenty to eighty days for most of the minor planets. As regards comets, especially those with small perihelion distances, it is necessary when they are near perihelion to use a much smaller interval. For economy in time, and in order to reduce the inevitable errors which accumulate during the extrapolation process, it is of great importance in practice to carry on the integration with a long interval. The question of augmenting the interval of integration can be solved in different ways. Firstly, one may consider the question of a more rational choice of special co-ordinates with which the formula (2) would be rigorous to the 8th or 10th order. Secondly, the problem of the systematic accumulation of the influence of higher orders (mainly of the 6th) during the extrapolation process can be solved. It is possible to find the systematic accumulation for any particular distant epoch by rejecting the quantity f, that is to say, by deliberately using a wrong formula. Thirdly, one can consider the question of computing the influence of higher orders from the co-ordinates of three epochs without having recourse to successive approximations. Lastly, the problem of the extrapolation of the perturbations themselves may be solved. It appears, besides, that in view of the smallness of the perturbations the influence of higher orders is very insignificant; that is why it is possible to conduct the extrapolation of perturbations with a longer interval than the extrapolation of the co-ordinates themselves, with the same accuracy. Moreover, the calculations in this case are carried out with a smaller number of figures, which is of great advantage in practice.

Passing over the first three methods, which are sufficiently dealt with in my other papers now ready for publication, I will describe in this paper a method of extrapolating the perturbations of the special co-ordinates, in the main similar to the method of Encke.

Simultaneously with the perturbed motion let us consider the unperturbed, and let us mark all the quantities of the unperturbed motion with the subscript o. It results from the formula (2) that the second difference of the co-ordinate $\bar{x}$ will be

$$\Delta^2\bar{x} = -\sigma\bar{x} + R + \tfrac{1}{12}\Delta^2 R + f \quad . \quad . \quad . \quad (6)$$

and similarly for the unperturbed motion

$$\Delta^2\bar{x} = -\sigma_0\bar{x}_0 + f_0 \quad . \quad . \quad . \quad . \quad (6^*)$$

If we denote by $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$ the perturbations of the special co-ordinates

$$\bar{x} - \bar{x}_0 = \bar{\xi}$$
$$\bar{y} - \bar{y}_0 = \bar{\eta}$$
$$\bar{z} - \bar{z}_0 = \bar{\zeta}$$

we find from (6) and (6*)

$$\Delta^2\bar{\xi} = -(\sigma\bar{x} - \sigma_0\bar{x}_0) + R + \tfrac{1}{12}\Delta^2 R + (f - f_0) \quad . \quad . \quad (7)$$

Denoting

$$\sigma_0 - \sigma = \Delta\sigma \quad . \quad . \quad . \quad . \quad . \quad (8)$$

we have

$$\Delta^2\bar{\xi} = -\sigma\bar{\xi} + R + \Delta\sigma\bar{x}_0 + \tfrac{1}{12}\Delta^2 R + (f - f_0) \quad . \quad . \quad (9)$$

If we compare the last formula with (6), we can see that the second difference of the perturbations is similar to the second difference of the perturbed co-ordinate $\bar{x}$. The formula (9) differs from the formula (6) by the supplementary term $\Delta\sigma\bar{x}_0$; moreover (and this is of great importance), the formula (9) is influenced only by the *difference* of the higher-order terms in the perturbed and unperturbed motion. Supposing the perturbations to be small and, therefore, that the elements of the osculating ellipse and initial ellipse do not differ much from each other, we see that the difference $(f - f_0)$ will be small and that the formula (9) thus admits a wider interval of integration. This will be of special importance for comets near perihelion, when the perturbations are, generally speaking, very small, but when the extrapolation of the co-ordinates themselves would have to be conducted with very short intervals.

The unperturbed co-ordinates $\bar{x}_0$, $\bar{y}_0$, $\bar{z}_0$ and the quantities $\bar{r}_0$, σ_0 are known from the unperturbed elements of a definite epoch of osculation, $\bar{r}_0$ and σ_0 being here defined by the formulæ

$$\left.\begin{aligned} \bar{r}_0 &= r_0(1 + \tfrac{1}{12}h) \qquad & h = \frac{k^2\omega^2}{r_0^{\,2}} \\ \sigma_0 &= \frac{h}{1 + \tfrac{1}{12}h} & \end{aligned}\right\} \quad . \quad . \quad . \quad (10)$$

The fundamental formula of extrapolation (9) may be transformed into

$$\bar{\xi}_3 = \bar{\xi}_2(2 - \sigma_2) - \bar{\xi}_1 + R_2 + \tfrac{1}{12}\Delta^2 R_2 + \Delta\sigma_2\bar{x}_0 + (f - f_0) \quad . \quad (11)$$

in which $\bar{x}_0$ refers to the unperturbed co-ordinate of the second epoch.

It is now necessary to compute the quantity $\Delta\sigma$. From formula (3) applied to the perturbed and unperturbed motions we obtain

$$\left(1-\frac{\Delta\sigma}{\sigma_0}\right)\left(1+\frac{\Delta\sigma}{12-\sigma_0}\right)^2=\frac{1}{\left(1+\frac{\Delta\bar{r}}{\bar{r}_0}\right)} \quad . \quad . \quad . \quad (12)$$

The difference $\Delta\bar{r}=\bar{r}-\bar{r}_0$ is connected with the quantity

$$\pi=\frac{1}{\bar{r}_0{}^2}[\bar{\xi}(\bar{x}_0+\tfrac{1}{2}\bar{\xi})+\bar{\eta}(\bar{y}_0+\tfrac{1}{2}\bar{\eta})+\bar{\zeta}(\bar{z}_0+\tfrac{1}{2}\bar{\zeta})] \quad . \quad . \quad (13)$$

by the relation

$$\pi=\frac{\Delta\bar{r}}{\bar{r}_0}+\frac{1}{2}\left(\frac{\Delta\bar{r}}{\bar{r}_0}\right)^2 \quad . \quad . \quad . \quad . \quad (14)$$

Thus we have the perturbations $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$ for any moment and also $\bar{x}_0$, $\bar{y}_0$, $\bar{z}_0$; from (13) we find π, and from (14) $\Delta\bar{r}:\bar{r}_0$, and lastly from (12) we find $\Delta\sigma$ when σ_0 is known.

In consideration of the smallness of the perturbations and the consequent smallness of the quantities π and $\Delta\sigma$, instead of a precise solution by the formulæ (12) and (14), we can make use of a development in series. Indeed, we can write

$$\Delta\sigma=3\sigma_0\pi(1+p) \quad . \quad . \quad . \quad . \quad (15)$$

where p is a small quantity depending on $\sigma_0=4\tau$ and π

$$\left.\begin{aligned}
p&=\alpha+\beta\pi+\gamma\pi^2+\delta\pi^3+\ \ldots\\
\alpha&=\frac{2\tau}{3(1-\tau)}\\
\beta&=-\tfrac{5}{2}\cdot\frac{3-\tau}{3(1-\tau)^3}(1-\tfrac{6}{5}\tau+\tfrac{3}{5}\tau^2)\\
\gamma&=\tfrac{35}{6}\cdot\frac{3-\tau}{3(1-\tau)^5}(1-\tfrac{16}{7}\tau+\tfrac{102}{35}\tau^2-\tfrac{12}{7}\tau^3+\tfrac{3}{7}\tau^4)\\
\delta&=-\tfrac{315}{24}\cdot\frac{3-\tau}{3(1-\tau)^7}(1-\tfrac{109}{63}\tau+\tfrac{47}{7}\tau^2-\tfrac{484}{63}\tau^3+\tfrac{37}{7}\tau^4-2\tau^5+\tfrac{1}{3}\tau^6)
\end{aligned}\right\} \quad . \quad (16)$$

Table I. gives the quantity p to five decimals as a function of the two arguments σ_0 and π. Having computed $\Delta\sigma$ by the formula (15), we find the perturbed value of σ by (8), and thus the fundamental formula of extrapolation (11) allows us to find from the given perturbations of two epochs the perturbation for the third one.

For the computation of the combined effect R from the planets by (4) it is necessary to transform the perturbed special co-ordinates into the ordinary ones, which is done by the formula

$$x=(\bar{x}_0+\bar{\xi})\left(1-\frac{\sigma}{12}\right) \quad . \quad . \quad . \quad . \quad (17)$$

TABLE I.

Values of p *according to Arguments* π *and* σ_0 *in Units of 5th Decimal.*

σ_0 / π.	0·00.	0·01	0·02.	0·03.	0·04.	0·05.	0·06.	0·07.	0·08.	0·09.	0
−0·010	+2560	+2736	+2914	+3092	+3272	+3452	+3634	+3816	+3999	+4183	+4
−0·009	2298	2474	2650	2828	3006	3186	3366	3547	3729	3913	4
−0·008	2038	2213	2388	2565	2742	2920	3100	3280	3461	3643	3
−0·007	1779	1953	2127	2303	2479	2657	2835	3014	3194	3375	3
−0·006	1521	1694	1868	2042	2218	2394	2571	2749	2928	3108	3
−0·005	1255	1436	1609	1783	1957	2132	2309	2486	2664	2843	3
−0·004	1009	1180	1352	1524	1698	1872	2048	2224	2401	2579	2
−0·003	755	925	1096	1268	1440	1613	1788	1963	2139	2316	2
−0·002	502	671	841	1012	1183	1356	1529	1703	1878	2054	2
−0·001	+ 251	419	587	757	928	1099	1272	1445	1619	1794	1
0·000	000	+ 167	335	504	673	844	1015	1187	1360	1534	1
+0·001	− 249	− 83	+ 84	252	420	590	760	932	1104	1277	1
0·002	− 498	− 332	− 166	+ 1	+ 168	337	506	677	848	1020	1
0·003	− 745	− 580	− 415	− 249	− 82	+ 85	254	423	593	764	
0·004	− 991	− 827	− 663	− 498	− 332	− 165	+ 2	+ 170	340	510	
0·005	−1235	−1073	− 910	− 745	− 580	− 415	− 248	− 81	+ 88	257	
0·006	−1479	−1318	−1155	− 992	− 828	− 663	− 497	− 331	− 163	+ 5	+
0·007	−1722	−1561	−1400	−1237	−1073	− 910	− 745	− 580	− 413	− 246	−
0·008	−1963	−1803	−1643	−1481	−1319	−1156	− 992	− 827	− 662	− 495	−
0·009	−2204	−2045	−1885	−1724	−1563	−1400	−1238	−1074	− 909	− 744	−
0·010	−2443	−2285	−2126	−1966	−1806	−1644	−1482	−1319	−1156	− 991	−

On the other hand, it is also possible to find the perturbations of the ordinary co-ordinates from the perturbations of special ones:

$$\xi = \bar{\xi} + \tfrac{1}{12}(\Delta\sigma\bar{x}_0 - \sigma\bar{\xi}) \quad . \quad . \quad . \quad . \quad (18)$$

which formula can also be employed for computing the initial value of $\bar{\xi}$ from ξ by making use of successive approximations.

The initial values of the perturbations for the first two moments are to be found from a development in powers of the time. If we denote by x_0 the unperturbed co-ordinate and by $x = x_0 + \xi$ the perturbed co-ordinate, we can write the fundamental equation of Encke's method

$$\omega^2\frac{d^2\xi}{dt^2} = -\left(\frac{k^2\omega^2 x}{r^3} - \frac{k^2\omega^2 x_0}{r_0^{\,3}}\right) + \mathrm{R} = -(\sigma\bar{x} - \sigma_0\bar{x}_0) + \mathrm{R} \quad . \quad (19)$$

or, after a transformation, similar to (7) and (9), we have

$$\omega^2\frac{d^2\xi}{dt^2} = \mathrm{R} - \sigma\bar{\xi} + \Delta\sigma\bar{x}_0 = \mathrm{F} \quad . \quad . \quad . \quad (19^*)$$

Let us represent the perturbation ξ near the osculation epoch t_0 in the form of a series

$$\xi = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + \ldots \quad . \quad . \quad (20)$$

It follows from the definition of the epoch of osculation that at $t=t_0$ we must have $\xi=\frac{d\xi}{dt}=0$, that is to say, from (20) we obtain two equations:

$$a_0+a_1t_0+a_2t_0^2+a_3t_0^3+\ .\ .\ .=0$$
$$a_1+2a_2t_0+3a_3t_0^2+4a_4t_0^3+\ .\ .\ .=0$$

Eliminating the coefficients a_0 and a_1 by means of these last equations, the formula (20) may be written thus:

$$\xi=(t-t_0)^2\{a_2+a_3(t+2t_0)+a_4(t^2+2tt_0+3t_0^2)\ .\ .\ .\ \} \quad . \quad (21)$$

Suppose that we know the right-hand side of the equation (19), namely, F_1, F_2, F_3 for three moments of time t_1, t_2, t_3 near the osculation epoch. In that case we have from (20)

$$\left.\begin{aligned} 2a_2+6a_3t_1+12a_4t_1^2&=\frac{1}{\omega^2}F_1\\ 2a_2+6a_3t_2+12a_4t_2^2&+\frac{1}{\omega^2}F_2\\ 2a_2+6a_3t_3+12a_4t_3^2&=\frac{1}{\omega^2}F_3 \end{aligned}\right\} \quad . \quad . \quad . \quad (22)$$

Should we choose the moments at equal intervals ω and suppose for the sake of simplicity $t_2=0$, $t_1=-\omega$ and $t_3=\omega$, $t_0=\omega\tau$, by use of the formula (22) the equation (21) can be transformed into the following final form:

$$\xi=\frac{(t-t_0)^2}{\omega^2}\left\{\frac{F_2}{2}+\frac{F_3-F_1}{12\omega}(t+2t_0)+\frac{F_1+F_3-2F_2}{24\omega^2}(t^2+2tt_0+3t_0^2)\right\} \quad (23)$$

Applying the last formula for three moments t_1, t_2, t_3, we find:

$$\left.\begin{aligned} \xi_1&=F_1(1,\ 1)+F_2(1,\ 2)+F_3(1,\ 3)\\ \xi_2&=F_1(2,\ 1)+F_2(2,\ 2)+F_3(2,\ 3)\\ \xi_3&=F_1(3,\ 1)+F_2(3,\ 2)+F_3(3,\ 3) \end{aligned}\right\} \quad . \quad . \quad . \quad (24)$$

where the coefficients are expressed in terms of

$$\tau=\frac{t_0-t_2}{\omega} \quad . \quad . \quad . \quad . \quad . \quad (25)$$

by the formulæ

$$\left.\begin{aligned} (1,\ 1)&=(3,\ 3)=\frac{(1-\tau^2)^2}{8} & (2,\ 2)&=\frac{\tau^2}{4}(2-\tau^2)\\ (2,\ 1)&=\frac{\tau^2}{24}(3\tau^2-4\tau) & (2,\ 3)&=\frac{\tau^2}{24}(3\tau^2+4\tau)\\ (3,\ 1)&=-\frac{(1-\tau)^2(1+2\tau-3\tau^2)}{24} & (1,\ 3)&=-\frac{(1+\tau)^2(1-2\tau-3\tau^2)}{24}\\ (1,\ 2)&=\frac{(1+\tau)^2(5+2\tau-3\tau^2)}{12} & (3,\ 2)&=\frac{(1-\tau)^2(5-2\tau-3\tau^2)}{12} \end{aligned}\right\} \quad (26)$$

Table II. gives the coefficients (1, 1), (2, 1) . . . (3, 3) as functions of the argument τ.

TABLE II.

Coefficients for computing the Perturbations of Initial Moments in Function of the Argument τ in Units of 5th Decimal.

τ+ve.	(1, 1)=(3, 3).	(1, 2).	(1, 3).	(2, 1).	(2, 2).	(2, 3).	(3, 1).	(3, 2).
0·000	+12500	+41667	−4167	0	+ 0	+ 0	−4167	+41667
0·025	12484	44197	4150	0	31	0	4152	39198
0·050	12438	46787	4100	− 2	125	2	4108	36796
0·075	12360	49433	4012	7	280	8	4040	34461
0·100	12251	52131	3882	16	498	18	3949	32198
0·125	+12112	+54877	−3708	− 30	+ 775	+ 36	−3838	+30007
0·150	11944	57666	3485	50	1112	62	3710	27892
0·175	11746	60496	3211	78	1508	101	3568	25853
0·200	11520	63360	2880	113	1960	153	3413	23893
0·225	11266	66254	2493	158	2467	222	3249	22014
0·250	+10986	+69173	−2034	− 212	+ 3027	+ 309	−3076	+20215
0·275	10681	72112	1511	275	3638	418	2898	18498
0·300	10351	75064	915	349	4298	551	2716	16864
0·325	9999	78025	− 242	433	5002	712	2531	15313
0·350	9625	80987	+ 512	527	5750	902	2346	13846
0·375	+ 9232	+83946	+1354	− 632	+ 6537	+1126	−2162	+12461
0·400	8820	86893	2287	747	7360	1387	1980	11160
0·425	8392	89824	3316	872	8216	1687	1802	9941
0·450	7950	92729	4446	1006	9100	2031	1629	8804
0·475	7496	95603	5683	1150	10008	2422	1462	7748
0·500	+ 7031	+98438	+7031	−1302	+10938	+2864	−1302	+ 6771
τ+ve.	(1, 1)=(3, 3).	(3, 2).	(3, 1).	(2, 3).	(2, 2).	(2, 1).	(1, 3).	(1, 2).

So far as it is possible to neglect the indirect terms in the formula (19) owing to the smallness of perturbations, the function F will be equal to R.

We suppose the perturbations of the special co-ordinates equal to the perturbations of the ordinary ones and compute the value of F by formula (19*). With the help of the corrected value of F we find, using (24), more approximate values of perturbations ξ, η, ζ, and lastly, by means of (18) we determine the perturbations of the special co-ordinates.

This method will now be illustrated by an example of the motion of the planet Erato (62) taken from the work of Oppolzer.*

In Table III. we give the special unperturbed co-ordinates of the planet, $\bar{x}_0$, $\bar{y}_0$, $\bar{z}_0$, to five decimals, computed from the elements.

* Th. R. v. Oppolzer, *Lehrbuch zur Bahnbestimmung der Kometen und Planeten*, B. ii., pp. 105 *et seq.*

TABLE III.

		N.	$\frac{\bar{x}_0}{\xi:2}$.	$\frac{\bar{y}_0}{\eta:2}$.	$\frac{\bar{z}_0}{\zeta:2}$.	$\frac{\bar{r}_0^2}{\pi\cdot 10^7}$.	$\frac{2-\sigma_0}{\Delta\sigma\cdot 10^7}$.
1875	ii. 24	1	−2·51804	−2·66282	+0·13867	13·4504	1·9903866
			− 3	+ 1	0	71	2·0
	i. 15	2	−2·75005	−2·43244	+0·14075	13·4993	1·9904389
			0	0	0	10	0·3
1874	xii. 6	3	−2·95575	−2·17879	0·14148	13·5036	1·9904434
			0	0	0	12	0·3
	x. 27	4	−3·13321	−1·90431	0·14086	13·4632	1·9904003
			− 4	+ 1	0	128	3·7
	ix. 17	5	−3·28059	−1·61155	0·13888	13·3787	1·9903091
			− 10	+ 4	0	414	12·1
	viii. 8	6	−3·39616	−1·30318	0·13556	13·2506	1·9901680
			− 21	+ 8	0	931	27·5
	vi. 29	7	−3·47837	−0·98200	0·13091	13·0805	1·9899753
			− 37	+ 13	0	1743	52·4
.	.	.	. .	. .	. .	. .	.
1872	xi. 6	22	−0·28468	2·76602	−0·05331	7·73475	1·9779090
			− 1249	+ 692	+ 13	59218	3881·3
	ix. 27	23	+0·15296	2·72169	−0·06600	7·43535	1·9765561
			− 1352	+ 900	+ 12	61013	4242·9
	viii. 18	24	+0·58702	2·61356	−0·07715	7·18124	1·9752955
			− 1421	+ 1144	+ 10	60988	4470·2
	vii. 9	25	+1·00656	2·44086	−0·08639	6·97842	1·9742061
			− 1443	+ 1419	+ 5	58844	4506·5

The epoch of osculation is 1874 December 26·0, Berlin M.T.:

$$M_0 = 180^\circ\ 40'\ 48''\cdot 9 \qquad \pi = 38^\circ\ 27'\ 17''\cdot 9$$
$$\log a = 0\cdot 4954793 \qquad \Omega = 125\ \ 42\ \ 39\cdot 7 \Big\}\ 1870\cdot 0.$$
$$\mu = 640''\cdot 89605 \qquad i = 2\ \ 12\ \ 23\cdot 9$$
$$\phi = 9^\circ\ 59'\ 14''\cdot 9$$

The quantities r_0^2, $(2-\sigma_0)$, and $\Delta\sigma$ are also given in Table III.

Table IV. gives the total combined effect of Jupiter and Saturn in units of the 7th decimal ($\omega = 40$), for the first three moments, computed by formula (4). These numbers are taken from Oppolzer's example. The initial values of perturbations for the first three moments near the osculation epoch are determined by the formula (24). In our case $\tau = +0\cdot 5$.

In the first approximation the perturbations are found by the formula (22), assuming F = R.

TABLE IV.

N.	$\xi.10^7$.	$\eta.10^7$.	$\zeta.10^7$.	$R_x.10^7$.	$R_y.10^7$.	$R_z.10^7$.
1	−658·3	+264·5	+10·1	−528·0	+209·9	−8·1
2	−75·6	+30·4	−1·2	−584·9	+235·2	−9·0
3	−78·2	+31·5	−1·2	−646·4	+259·7	−9·7

If we assume the perturbations of the real co-ordinates to be equal to those of the special ones, we find the corrected values of the second derivatives from the perturbations by formula (19).

TABLE V.

N.	$\frac{\bar{x}_0}{\bar{\xi}:2}$.	$\frac{\bar{y}_0}{\bar{\eta}:2}$.	$\frac{\bar{z}_0}{\bar{\zeta}:2}$.	$\frac{\bar{r}_0^2}{\pi\ 10^7}$.	$\frac{2-\sigma_0}{\Delta\sigma.10^8}$.	$R_x.10^7$.	$F_x.10^7$.	$R_y.10^7$.	$F_y.10^7$.	$R.10^7$.	F_z.
1	−2 51804	−2·66282	+0·13867	13·4504	1·9903866	−528·0	−526·7	+209·9	202·1	−8·1	–
	− 3	+ 1	0	71	20	− 5·0		− 5·3		+0·3	
						+ 6·3		− 2·5		+0·1	
2	−2·75005	−2·43244	+0·14075	13·4993	1·9904389	−584·9	−585·0	+235·2	234·2	−9·0	–
	0	0	0	10	03	− 0·8		− 0·7		0·0	
						+ 0·7		− 0·3		0·0	
3	−2·95575	−2·17879	+0·14148	13·5036	1·9904434	−646·1	−646·3	+259·7	258·8	−9·7	–
	0	0	0	12	03	− 0·9		− 0·6		0·0	
						+ 0·7		− 0·3		0·0	

With the improved values of F we recompute the perturbations of ordinary co-ordinates ξ, η, ζ by the formula (24), then from (18), with the values given in Table V., we find the perturbations of the special co-ordinates $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$.

TABLE VI.

N.	$\xi.10^7$.	$\bar{\xi}.10^7$.	$\eta.10^7$.	$\bar{\eta}.10^7$.	$\zeta.10^7$.	$\bar{\zeta}.10^7$.
1	−658·3	−658·4	263·0	263·6	−10·1	−10·1
2	−75·6	−75·6	30·4	30·5	−1·2	−1·2
3	−78·2	−78·2	31·4	31·5	−1·2	−1·2

Having thus obtained the perturbations for the first two moments, we proceed with the extrapolation by formula (11), in the first place finding the perturbations for the third moment as a numerical check. The computation of the perturbations is given in Tables VII. and III. The combined effects were borrowed from Oppolzer's example, and are corrected for second differences.

The perturbations are computed successively by (13), (15), and (16), and the multiplier p is taken from Table I. as a function of the two arguments π and σ_0.

In the same Table VII. we give the final values of the perturbations ξ, η, ζ of the ordinary rectangular ecliptic co-ordinates of the planet

Erato (62), computed by formula (18). The differences $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$ from Oppolzer's results are given in the table.

TABLE VII.

N.	$\frac{R_x}{\Delta\sigma \cdot \bar{x}_0}$.	$\frac{\bar{\xi} \cdot 10^7}{\xi - \bar{\xi}}$.	$\frac{\xi \cdot 10^7}{\Delta\xi}$.	$\frac{R_y}{\Delta\sigma \bar{y}_0}$.	$\frac{\bar{\eta} \cdot 10^7}{\eta - \bar{\eta}}$.	$\frac{\eta \cdot 10^7}{\Delta\eta}$.	$\frac{R_z}{\Delta\sigma \cdot \bar{z}_0}$.	$\frac{\bar{\zeta} \cdot 10^7}{\zeta - \bar{\zeta}}$.	$\frac{\zeta \cdot 10^7}{\Delta\zeta}$.	$\frac{2-\sigma}{\sigma : 12}$.
1	− 528·0	− 658·4	− 658·3	+209·9	263·6	263·3	− 8·1	−10·1	−10·0	1·9903868
	− 5·0	+ 0·1	0	− 5·3	−0·6	0	+ 0·3	0·1	0	0·00080110
2	− 585·3	− 75·6	− 75·6	235·1	30·5	30·4	− 9·0	− 1·2	− 1·2	1·9904389
	− 0·8	+ 0·0	0	− 0·7	−0·1	0	0	0	0	79676
3	− 646·4	− 78	− 78	259·7	31	31	− 9·7	− 1	− 1	1·9904434
	− 0·9	0	0	− 0·7	0	+1	0	0	0	79638
4	− 711·2	− 727	− 727	281·2	290	289	− 10·0	− 10	− 10	1·9904007
	− 11·6	0	0	− 7·0	− 1	+2	0·5	0	− 1	79994
5	− 778·3	− 2092	− 2093	298·2	820	818	− 9·7	− 29	− 29	1·9903103
	− 39·7	− 1	0	− 19·5	− 2	+2	+ 1·7	0	− 1	80747
6	− 844·8	− 4255	− 4259	307·6	1621	1617	− 8·7	− 56	− 56	1·9901708
	− 93·4	− 4	0	− 35·8	− 4	+3	+ 3·7	0	− 1	81910
7	− 906·9	− 7314	− 7323	306·8	2678	2671	− 6·9	− 87	− 86	1·9899805
	− 182·2	− 9	0	− 51·5	− 7	+3	+ 6·9	+ 1	− 1	83496
.	.	.	.	.	.	.	.	.	.	.
22	− 194·7	−249874	−249454	− 36·6	138459	139103	8·7	2636	2614	1·9782971
	−1104·9	+ 360	− 10	10735·8	+644	−1	−206·9	− 22	+ 2	180857
23	− 156·4	−270323	−269750	− 39·0	179996	180613	7·3	2452	2424	1·9769804
	+ 649·0	+ 573	− 9	11547·8	+617	0	−280·1	− 28	+ 2	191830
24	− 124·7	−284117	−283324	− 42·0	228898	229409	6·0	1939	1903	1·9757426
	+2624·0	+ 793	− 7	11683·9	511	0	−344·9	− 36	+ 2	202146
25	− 98·9	−288520	−287533	− 45·5	283889	284206	4·9	1040	1005	1·9746568
	+4536·6	+ 987	− 4	11000·1	+317	+1	−389·3	− 35	+ 2	21193

The divergence at the end of twenty-five steps of extrapolation does not exceed ten units of the 7th decimal, which are due to cumulative errors during the process of numerical integration.

The method here described can be applied not only to the computation of the perturbations, in comparison with elliptic motion, but generally for finding the differences between two perturbed motions. For instance, one can in this way take into account separately the perturbations caused by Saturn, or the systematic effect of higher-order terms neglected in the extrapolation of perturbed co-ordinates.

Petrograd,
The Astronomical Observatory
of the Petrograd University:
1924 *January.*

Note on the Numerical Integration of $\frac{d^2x}{dt^2}=f(x, t)$.

By J. Jackson, M.A., D.Sc.

After the discovery of the eighth satellite of Jupiter, Cowell introduced numerical integration into dynamical astronomy as a method for taking account of all the forces, including the action of the central body. This method he applied later to the investigation of the motion of Halley's Comet between 1759 and 1910. This work of Cowell's has led other workers to deal with the subject of numerical integration, as it affords a ready solution to problems otherwise intractable. Cowell, instead of using the formulæ for numerical integration which had been left in a practically perfect state by Gauss,* reinvestigated the problem and introduced formulæ for changing the second differential coefficients into second differences, so that the integral was obtained at once by a simple double summation. After the work had been done, Cowell found out the disadvantages of the method he introduced, and incorporated the long-established formula in the first of two suggestions for future computations, which are given at the end of the essay by himself and Crommelin on the motion of Halley's Comet, published as an appendix to the Greenwich Observations for 1909. As these suggestions do not appear to be well known they are repeated here.

(1) In place of the fundamental formula

$$x_2 = \mathrm{X} + \tfrac{1}{12}\mathrm{X}_2 - \tfrac{1}{240}\mathrm{X}_4 + \tfrac{1}{1951}\mathrm{X}_6 \ . \ . \ .$$

it would have been far better to use

$$x = \mathrm{X}_{-2} + \tfrac{1}{12}\mathrm{X} - \tfrac{1}{240}\mathrm{X}_2 + \tfrac{1}{1951}\mathrm{X}_4. \ . \ . \ .$$

(2) Short time intervals ought to be employed, as errors in that case are far easier to detect. The intervals in the *Ast. Ges.* essay were far too long, and those in the revised calculations of this appendix not too short. The intervals should be 1 day at perihelion, and may be doubled at 40, 80, 160 days, and so on from perihelion. The interval should not be changed during a close approach to a planet.

Of these suggestions (1) is a return to a long-established formula and (2) is a point of practical importance, the truth of which is borne home with greater and greater force the more one computes. A numerical mistake made at any point may vitiate all subsequent work. Professor Noumerov has recently introduced "special co-ordinates" in

* The formulæ for numerical integration go right back to the beginnings of the calculus. Formulæ were developed by Newton, Gregory, and others in the seventeenth and eighteenth centuries. The formulæ advocated here were known to Gauss early in the nineteenth century. The subject was carefully discussed by Encke in the Berlin *Jahrbuch* for 1837, and is given in great detail by Oppolzer at the beginning of the second volume of his *Lehrbuch zur Bahnbestimmung der Kometen und Planeten.*